THE SU(1, 1) PROPAGATOR AS A PATH INTEGRAL OVER NONCOMPACT GROUPS

G. JUNKER and M. BÖHM

Physikalisches Institut der Universität Würzburg, Am Hubland, 8700 Würzburg, FRG

Received 5 May 1986; accepted for publication 14 July 1986

The path integral on a noncompact group manifold is constructed. Using the Fourier decomposition on SU(1, 1) the corresponding propagator is calculated. An application is made for the modified Pöschl–Teller potential, where the energy eigenvalues and the normalized wavefunctions of bound and scattering states are found simultaneously.

In recent years, many potentials obeying a certain dynamical symmetry have been added to the list of path-integrable systems. Employing completeness and orthogonality of the corresponding group representations has been found to be an important tool for the integration over paths. For example, the SO(4) dynamical symmetry of the Coulomb [1] and the dyonium problem [2] has been utilized via a nonlinear coordinate transformation. By a different technique, various systems with dynamical SU(2) symmetry have become solvable, too [3–5]. From that point of view, the evaluation of the path integral over group manifolds is important. Up to now only bound-state problems related to compact groups have been discussed. In order to solve scattering problems one has to consider the path integral over noncompact groups.

One of the simplest of this type is SU(1, 1). Recently this group has attracted much attention in the algebraic approach to scattering theory [6]. The purpose of this paper is to perform the path integral over the SU(1, 1) manifold. First we will give a review of this group and its representations to derive an explicit expression for the Fourier analysis on SU(1, 1). Then the Feynman propagator is constructed. Expanding the short-time propagator into SU(1, 1) matrix elements, the integration over the paths can be performed. Finally we will apply this technique to the one-dimensional modified Pöschl–Teller potential.

The spinor representation of SU(1, 1) has been discussed by Bargmann [7]. Usually the elements are parametrized by eulerian "angles"

$$g(\varphi, \tau, \psi) = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \begin{pmatrix} \cosh(\tau/2) & \sinh(\tau/2) \\ \sinh(\tau/2) & \cosh(\tau/2) \end{pmatrix} \begin{pmatrix} e^{-i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix}, \tag{1}$$

$$0 \leqslant \varphi < 2\pi, \quad 0 \leqslant \tau < \infty, \quad 0 \leqslant \psi < 4\pi.$$

With these parameters the invariant volume element is given by $dg = \sinh\tau \, d\tau \, d\varphi \, d\psi/16\pi^2$. The unitary irreducible representations of the fundamental series are [7]

$$\begin{aligned} D_l^\sigma: \quad & l = -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \dots \quad && m = l+1, l+2, \dots, && \text{for } \sigma = +, \\ & && m = -l-1, -l-2, \dots, && \text{for } \sigma = -, \\ & l = -\tfrac{1}{2} + is, && s \geqslant 0, \ m = 0, \pm 1, \pm 2, \dots, && \text{for } \sigma = 0, \\ & && s > 0, \ m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \dots, && \text{for } \sigma = \tfrac{1}{2}. \end{aligned} \tag{2}$$

According to eq. (1) the matrix elements are given by the multiplier representation

$$d_{mn}^{l,\sigma}(g) = e^{-im\varphi} e^{-in\psi} V_{mn}^{l,\sigma}(\tau). \tag{3}$$

0375-9601/86/$03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

Actually the Bargmann function $V^{l,\sigma}_{mn}(\tau)$ may be understood as an analytical continuation in parameter θ and index j of Wigner's $d^j_{mn}(\theta) \in \mathrm{SU}(2)$. Explicitly we have for $m \geqslant n$

$$V^{l,+}_{mn}(\tau) = \frac{1}{(m-n)!}\left(\frac{\Gamma(1+m+l)\Gamma(m-l)}{\Gamma(1+n+l)\Gamma(n-l)}\right)^{1/2}[\cosh(\tau/2)]^{-m-n}[\sinh(\tau/2)]^{m-n}$$
$$\times {}_2F_1(1-n+l,\ -n-l;\ 1+m-n;\ -\sinh^2(\tau/2)),$$
$$V^{l,-}_{mn}(\tau) = \frac{1}{(m-n)!}\left(\frac{\Gamma(1-n+l)\Gamma(-n-l)}{\Gamma(1-m+l)\Gamma(-m-l)}\right)^{1/2}[\cosh(\tau/2)]^{m+n}[\sinh(\tau/2)]^{m-n}$$
$$\times {}_2F_1(1+m+l,\ m-l;\ 1+m-n;\ -\sinh^2(\tau/2)). \tag{4}$$

Note that $V^{l,\sigma}_{mn}(\tau) = (-1)^{m-n} V^{l,\sigma}_{nm}(\tau)$. For the continuous series $V^{-1/2+\mathrm{i}s,\sigma}_{mn}(\tau)$ may be obtained by analytical continuation in the index l. As was shown by Bargmann, the Hilbert space of square integrable functions on SU(1, 1) is spanned by the matrix elements of the continuous and the discrete class for $l \geqslant 0$.

For the Fourier analysis [8]

$$\hat{f}(\lambda) = \int f(g) D^{\lambda}(g^{-1})\,\mathrm{d}g, \qquad f(g) = \sum\!\!\!\!\!\!\int \mathrm{d}\lambda\, \mathrm{Tr}\{\hat{f}(\lambda) D^{\lambda}(g)\} d_{\lambda}, \tag{5}$$

only these have to be considered. The "dimensions" d_λ for SU(1, 1) have the values

$$\begin{aligned} d_\lambda &= 2l+1, && \text{for } l = 0, \tfrac{1}{2}, 1, \ldots, \\ &= 2s\ \tanh[\pi(s+\mathrm{i}\sigma)], && \text{for } l = -\tfrac{1}{2} + \mathrm{i}s. \end{aligned} \tag{6}$$

Together with (5) the explicit Fourier decomposition on SU(1, 1) is now given by

$$f(g) = \sum_{\sigma}\left[\left(\sum_{2l=0}^{\infty}(2l+1) + \int_0^{\infty}\mathrm{d}s\ 2s\ \tanh[\pi(s+\mathrm{i}\sigma)]\right) a^{\sigma}_{mn}(l)\, d^{l,\sigma}_{mn}(g)\right], \tag{7a}$$

with

$$a^{\sigma}_{mn}(l) = \int_{\mathrm{SU}(1,1)} f(g)[d^{l,\sigma}_{mn}(g)]^{*}\,\mathrm{d}g. \tag{7b}$$

In order to construct the Feynman propagator we make use of an isomorphism between the SU(1, 1) manifold and a four-dimensional hyperboloid given by

$$(\boldsymbol{r}, \boldsymbol{r}) = (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 = \text{const.} \tag{8}$$

We introduce the following coordinates for the pseudo-euclidean space defined by (8).

$$\begin{aligned} x^1 &= r\cosh(\tau/2)\cos[(\varphi+\psi)/2], && 0 \leqslant \tau < \infty, \\ x^2 &= r\cosh(\tau/2)\sin[(\varphi+\psi)/2], && 0 \leqslant \varphi < 2\pi, \\ x^3 &= r\sinh(\tau/2)\cos[(\varphi-\psi)/2], && 0 \leqslant \psi < 4\pi, \\ x^4 &= r\sinh(\tau/2)\sin[(\varphi-\psi)/2]. && \end{aligned} \tag{9}$$

With the given metric the short-time action for a free particle in the usual sliced-time basis is

$$\tilde{S}_j = (m/2\epsilon)\left[(\Delta x^1_j)^2 + (\Delta x^2_j)^2 - (\Delta x^3_j)^2 - (\Delta x^4_j)^2\right]. \tag{10}$$

This form requires some modification of the standard path-integral formalism derived by Langguth and Inomata [9]. For the compact subspace, the x^1–x^2 plane, the mass still has a small positive imaginary part, $M = m + \mathrm{i}\eta$ $(\eta > 0)$. For the noncompact subspace, however, one needs a small negative imaginary term $M = m - \mathrm{i}\eta$, in order to have well-defined integrals. Moreover, the normalization factors of the Feynman ansatz have to be chosen in the following way

$$K(\boldsymbol{r}_\mathrm{b}, \boldsymbol{r}_\mathrm{a}; t_\mathrm{b} - t_\mathrm{a}) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{e}^{(\mathrm{i}/\hbar)\tilde{S}_j} \prod_{j=1}^{N} \frac{m}{2\pi\mathrm{i}\hbar\epsilon} \frac{\mathrm{i}m}{2\pi\hbar\epsilon} \prod_{j=1}^{N-1} \mathrm{d}^4 r_j . \tag{11}$$

Expressing $\tilde{S}_j$ in the coordinates (9) and constraining onto the hyperboloid $r = 1$ in the usual way [4], yields the following ansatz for the SU(1, 1) propagator,

$$K(\boldsymbol{e}_\mathrm{b}, \boldsymbol{e}_\mathrm{a}; t_\mathrm{b} - t_\mathrm{a}) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{e}^{(\mathrm{i}/\hbar)S_j} \prod_{j=1}^{N} \left(\frac{m}{2\pi\hbar\epsilon}\right)^{1/2} \frac{\mathrm{i}m}{2\pi\hbar\epsilon} \prod_{j=1}^{N-1} \tfrac{1}{8} \sinh\tau_j \,\mathrm{d}\tau_j \,\mathrm{d}\varphi_j \,\mathrm{d}\psi_j ,$$

$$S_j = (m/\epsilon)\left[1 - (\boldsymbol{e}_j, \boldsymbol{e}_{j-1})\right], \tag{12}$$

where $\boldsymbol{e}$ is a unit vector on the pseudosphere $(\boldsymbol{r}, \boldsymbol{r}) = 1$.

Identifying the coordinates (9) with the parameters (1), defining $g_j = g(\varphi_j, \tau_j, \psi_j)$ and $\hat{g}_j = (\hat{\varphi}_j, \hat{\tau}_j, \hat{\psi}_j) = g_j g_{j-1}^{-1}$, the action may be expressed by

$$S_j = (m/\epsilon)\left[1 - \tfrac{1}{2}\,\mathrm{Tr}(\hat{g}_j)\right] = (m/\epsilon)\left\{1 - \cosh(\hat{\tau}_j/2)\cos\left[(\hat{\varphi}_j + \hat{\psi}_j)/2\right]\right\}. \tag{13}$$

Eqs. (12) and (13) are the basis for the expansion of the short-time propagator into SU(1, 1) matrix elements. Namely using the generating functions for Bessel functions

$$\exp\{-\mathrm{i}z\cosh(\tau/2)\cos[(\varphi+\psi)/2]\} = \sum_{2p=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}p(\varphi+\psi)}(-\mathrm{i})^{2p} J_{2p}(z\cosh(\tau/2)) \tag{14}$$

we obtain the Fourier coefficients

$$a^{\sigma}_{mn}(l) = \tfrac{1}{2}(-\mathrm{i})^{2m}\delta_{mp}\delta_{np} I(z), \tag{15}$$

with the remaining integral

$$I(z) = \int_0^{\infty} J_{2m}(z\cosh(\tau/2))(\cosh(\tau/2))^{2m}\, {}_2F_1(1+l+m,\, m-l;\, 1;\, -\sinh^2(\tau/2)) \sinh\tau \,\mathrm{d}\tau. \tag{16}$$

Expressing the Bessel function in terms of Meier's G-function, the integral can be performed for Re $l \geqslant -\frac{1}{2}$, using ref. [10]. After some algebraic manipulations one finds

$$I(z) = \frac{4\mathrm{i}^{2m}}{\pi z}\left[(-1)^{2m} K_{2l+1}(z\,\mathrm{e}^{-\pi\mathrm{i}/2}) + K_{2l+1}(z\,\mathrm{e}^{\pi\mathrm{i}/2})\right]. \tag{17}$$

$K_\nu(x)$ is the modified Bessel function of the third kind. For large z and Re $z > 0$ (Re $z < 0$) in the discrete (continuous) case the coefficients (15) are

$$a^{\sigma}_{mn}(l) \sim \frac{1}{2\pi^2}\left(\frac{2\pi\mathrm{i}}{z}\right)^{1/2} \frac{2\pi}{\mathrm{i}z}\,\mathrm{e}^{-\mathrm{i}z} \exp\left(-\mathrm{i}\frac{(2l+1)^2 - \frac{1}{4}}{2z}\right)\delta_{mn}. \tag{18}$$

Together with eq. (8) we find the asymptotic relation for small ϵ and $M = m \pm i\eta$, respectively:

$$\exp\left(\frac{iM}{\hbar\epsilon}\left[1 - \tfrac{1}{2}\,\mathrm{Tr}(g)\right]\right) \sim \frac{1}{2\pi^2}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2}\frac{2\pi\hbar\epsilon}{im}$$
$$\times \sum_\sigma\left[\left(\sum_{2l=0}^{\infty}(2l+1) + \int_0^\infty ds\, 2s\, \tanh[\pi(s+i\sigma)]\right)\chi_l^\sigma(g)\, e^{-(i/\hbar)E_l\epsilon}\right], \tag{19}$$

where

$$\chi_l^\sigma(g) = \mathrm{Tr}\, D_l^\sigma(g), \qquad E_l = (\hbar^2/2m)\left[(2l+1)^2 - \tfrac{1}{4}\right]. \tag{20}$$

Using the orthogonality of characters

$$\int_{SU(1,1)} \chi_\lambda^\sigma(g_j g_{j-1}^{-1})\chi_{\lambda'}^{\sigma'}(g_{j+1}g_j^{-1})\, dg_j = \delta_{\sigma\sigma'}\delta(\lambda-\lambda')\chi_\lambda^\sigma(g_{j+1}g_{j-1}^{-1})/d_\lambda, \tag{21}$$

the resulting SU(1, 1) propagator is expressed in terms of group characters

$$K(\boldsymbol{e}_b, \boldsymbol{e}_a; t_b - t_a) = \frac{1}{2\pi^2}\sum_\sigma\left(\sum_{2l=0}^{\infty}(2l+1)\exp[-(i/\hbar)E_l(t_b-t_a)]\chi_l^\sigma(g_b g_a^{-1})\right.$$
$$\left. + \int_0^\infty ds\, 2s\, \tanh[\pi(s+i\sigma)]\exp\left[-(i/\hbar)E_{-\frac{1}{2}+is}(t_b-t_a)\right]\chi_{-1/2+is}^\sigma(g_b g_a^{-1})\right). \tag{22}$$

With eq. (22) we have completed the calculation of the propagator on the SU(1, 1) manifold, having a continuous and a discrete spectrum.

As an example, we would like to apply this treatment to the one-dimensional modified Pöschl–Teller potential (mPT)

$$V(x) = \frac{\hbar^2}{2m}\left(\frac{\mu^2 - \frac{1}{4}}{\sinh^2 x} - \frac{\nu^2 - \frac{1}{4}}{\cosh^2 x}\right), \quad 0 < x < \infty. \tag{23}$$

This problem is also of interest for the path integration over the SO(p, q) manifold [11]. The difference between (23) and the usual problem is that hyperbolic functions replace the trigonometric ones. So the mPT may be understood, in some way, to be the analytical continuation of the ordinary one, having SU(1, 1) instead of SU(2) as dynamical symmetry. Viewing SU(1, 1) as the continuation of SU(2), the mPT can be solved in an analogous way [4,5].

Putting $x = \tau/2$ one finds for the Feynman kernel

$$K^-(x_b, x_a; t_b - t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N} e^{(i/\hbar)S_j}\prod_{j=1}^{N}\left(\frac{im}{2\pi\hbar\epsilon}\right)^{1/2}\prod_{j=1}^{N-1}\tfrac{1}{2}\, d\tau_j,$$

$$S_j = (m/\epsilon)\left[1 - \cosh(\Delta\tau_j/2)\right] + \frac{\hbar^2}{2m}\left(\frac{\mu^2-\frac{1}{4}}{\sinh(\tau_j/2)\sinh(\tau_{j-1}/2)} - \frac{\nu^2-\frac{1}{4}}{\cosh(\tau_j/2)\cosh(\tau_{j-1}/2)} - \frac{1}{4}\right)\epsilon. \tag{24}$$

Here K^- is the usual propagator with ϵ replaced by $-\epsilon$. Eq. (24) may be transformed into that of the

SU(1, 1) propagator (12) by introducing two angular variables $\varphi \in [0, 2\pi]$ and $\psi \in [0, 4\pi]$. For this procedure one has to restrict μ and ν to positive integers [3–5]. The result is

$$K^{-}(x_b, x_a; t_b - t_a) = \tfrac{1}{4}(\sinh \tau_a \sinh \tau_b)^{1/2} \int_0^{2\pi}\int_0^{4\pi} Q(\boldsymbol{e}_b, \boldsymbol{e}_a; t_b - t_a)\, \mathrm{d}\psi_b\, \mathrm{d}\varphi_b, \tag{25}$$

with

$$Q(\boldsymbol{e}_b, \boldsymbol{e}_a; t_b - t_a) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{e}^{(\mathrm{i}/\hbar)\tilde{S}_j} \prod_{j=1}^{N} \left(\frac{m}{2\pi \mathrm{i}\hbar\epsilon}\right)^{1/2} \frac{\mathrm{i}m}{2\pi\hbar\epsilon} \prod_{j=1}^{N-1} \tfrac{1}{8} \sinh \tau_j\, \mathrm{d}\tau_j\, \mathrm{d}\varphi_j\, \mathrm{d}\psi_j,$$

$$\tilde{S}_j = (m/\epsilon)\left[1 - \tfrac{1}{2}\operatorname{Tr}(\hat{g}_j)\right] + \tfrac{1}{2}\hbar(\nu+\mu)\, \Delta\varphi_j + \tfrac{1}{2}\hbar(\nu-\mu)\, \Delta\psi_j - \hbar^2\epsilon/8m. \tag{26}$$

The complete propagator is found by making use of eqs. (19) and (21). After reversing time we have

$$K(x_b, x_a; t_b - t_a) = \sum_{l=\sigma}^{(\nu-\mu)/2-1} \exp[-(\mathrm{i}/\hbar)E_l(t_b - t_a)]\, \psi_l(x_b)\psi_l^*(x_a) + \int_0^{\infty} \mathrm{d}k\, \exp[-(\mathrm{i}/\hbar)E_k(t_b - t_a)]\, \phi_k(x_b)\phi_k^*(x_a), \tag{27}$$

where

$$E_l = -(\hbar^2/2m)(2l+1)^2, \qquad \psi_l(x) = [(2l+1)\sinh 2x]^{1/2} V^{l,+}_{(\nu+\mu)/2,(\nu-\mu)/2}(2x), \tag{28a}$$

$$E_k = \hbar^2 k^2/2m, \qquad \phi_k(x) = \left[\tfrac{1}{2}k \tanh[\pi(\tfrac{1}{2}k + \mathrm{i}\sigma)] \sinh 2x\right]^{1/2} V^{-1/2+\mathrm{i}k/2,\sigma}_{(\nu+\mu)/2,(\nu-\mu)/2}(2x), \tag{28b}$$

with $\sigma = 0$ (1/2) for $\mu + \nu$ even (odd). This result is in agreement with that obtained earlier for $\mu = 1/2$ [3,6]. Since the propagator (11) has the property

$$\lim_{t_b \to t_a} K(\boldsymbol{r}_b, \boldsymbol{r}_a; t_b - t_a) = \delta(\boldsymbol{r}_b - \boldsymbol{r}_a),$$

it is ensured that the wavefunctions (28) are correctly normalized. For $\nu < \mu + 2(\sigma + 1)$ the sum in (27) vanishes, i.e. there are no bound states. Finally, in the limit $x \to \infty$ we find from eq. (28b), $\phi_k(x) \sim A\, \mathrm{e}^{\mathrm{i}kx} + B\, \mathrm{e}^{-\mathrm{i}kx}$ with

$$\frac{A}{B} = 4^{-\mathrm{i}k} \frac{\Gamma(\mathrm{i}k)\Gamma((1+\mu-\nu)/2 - \mathrm{i}k/2)\Gamma((1+\mu+\nu)/2 - \mathrm{i}k/2)}{\Gamma(-\mathrm{i}k)\Gamma((1+\mu-\nu)/2 + \mathrm{i}k/2)\Gamma((1+\mu+\nu)/2 + \mathrm{i}k/2)}, \tag{29}$$

completing our discussion of the mPT.

The same technique may be applied to the modified nonsymmetric Rosen–Morse oscillator.

The authors would like to thank Professor A. Inomata for bringing this problem to their attention.

References

[1] I.H. Duru and H. Kleinert, Phys. Lett. B 84 (1979) 185;
R. Ho and A. Inomata, Phys. Rev. Lett. 48 (1982) 321.
[2] G. Junker and A. Inomata, preprint State Univ. of New York at Albany (1986);
H. Kleinert, preprint University of California, San Diego (1986).
[3] A. Inomata and M.A. Kayed, J. Phys. A 18 (1985) 235.
[4] G. Junker and A. Inomata, in: Proc. Conf. on Path integrals form meV to MeV (World Scientific, Singapore), to be published.
[5] A. Inomata, G. Junker and R. Wilson, preprint State Univ. of New York at Albany (1986).
[6] Y. Alhassid, F. Gürsey and F. Iachello, Ann. Phys. 148 (1983) 346; 167 (1986) 181.

[7] V. Bargmann, Ann. Math. 48 (1947) 568 (Bargmann's notation is $k = l + 1$).
[8] R. Hermann, Fourier analysis on groups and partial wave analysis (Benjamin, New York, 1969) p. 191.
[9] W. Langguth and A. Inomata, J. Math. Phys. 20 (1979) 499.
[10] I.S. Gradsteyn and I.M. Ryzhik, Tables of integrals, series and products (Academic Press, New York, 1969) p. 903.
[11] G. Junker, Diploma thesis at the Universität Würzburg (1986).